State scarring by "ghosts" of periodic orbits

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In this paper we discover a generic structure in the eigenfunctions of quantum billiards, namely, scarring by families of stable periodic orbits in a nonchaotic system. Our study is conducted on two different triangular billiards, one an ergodic system and the other a "pseudointegrable" billiard. Surprisingly, we detect scars in regions which contain no periodic orbits. The periodic orbits responsible for scarring reside in a "neighboring" triangle. Such orbits show a more complex phase space structure than the "bouncing ball" trajectories of the stadium billiard. While diffuse nodal structure is usually the antithesis of scarring, we show that in some eigenstates it is supported by extensive families of stable periodic orbits.

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I. INTRODUCTION

The quantum mechanics of classically integrable systems is fairly well understood. Exact solutions to the Schrödinger equation and fully developed semiclassical approximations have been known for many years. During the past few decades, however, the "chaos revolution" of classical mechanics has exposed new dynamical behavior which goes far beyond the simple integrability of graduate level textbooks. Though a full understanding of the quantum properties of classically chaotic systems is not yet available, many fundamental discoveries have nonetheless been made in the past years. For example, common features have been discovered in the nearest neighbor spacing distributions of energy levels for classically nonintegrable systems [1]. The semiclassical approach by Gutzwiller [2] has proved to be very successful in showing the importance of classical periodic orbits for determining the structure of the quantum spectrum. However, understanding the characteristics of eigenfunctions of generic systems has proven to be a much more elusive task.

Quantum billiards are a particularly rewarding paradigm for the investigation of the properties of the eigenfunctions for many reasons: the numerical solution to the two-dimensional Schrödinger equation for a free particle with Dirichlet boundary conditions is a relatively easy task [3–5]. Moreover, billiards contain a cornucopia of diverse classical motions, offering the opportunity to study the quantum eigenstates of many classical regimes. Finally, billiards are especially suited to a visually effective display of the eigenfunctions.

For many years the phase space representation (via either the Wigner or the Husimi transform) of any eigenfunctions of a classically nonintegrable system has been conjectured to be uniformly distributed on the classical energy hypersurface of the system [6] corresponding to the eigenvalue of that state. In quantum billiards, this conjecture predicts eigenfunctions to be evenly spread in configuration space. The first evidence of the fallacy in this conjecture came with the diagonalization of the stadium billiard [7], a system known to be classically chaotic [8]. There, classical periodic orbits were observed to play a significant role in the shaping of the eigenfunctions. Some eigenstates were seen to concentrate along *isolated* periodic orbits. Heller [9] provided the first theoretical interpretation of the phenomenon, which he dubbed "scars." Other eigenfunctions were seen to be localized along families of very simple orbits (the so called "bouncing ball" states) which shuttle back and forth between the straight sides of the stadium. Their existence can be interpreted in terms of the Born-Oppenheimer approximation [10]. Bouncing ball eigenstates were also observed in the $(\pi/3)$-rhombus billiard [11]. Most recently, further theoretical understanding of scars has come from semiclassical calculations [12,13]. Berry [13] showed that scars are best understood in the phase space of the system, rather than its configuration space only. This phase space approach by Berry was later extended to include the case of billiards [14]. It is noteworthy that all the mentioned theoretical arguments assume the orbit scarring some eigenstate to be isolated: this is the typical case in chaotic systems.

Triangular billiards are a special case of polygonal billiards. Polygonal billiards can be either exactly integrable (only in a few exceptional cases), or "pseudointegrable" (when their angles are rational multiples of π), or ergodic (when their angles are irrational multiples of π) [15]. Polygonal billiards cannot be chaotic (they are A-integrable systems [16]), and they do not have positive Lyapunov exponents. They can support only stable orbits, with the negligible exception of those orbits that hit the boundary next to some vertex of the polygon. The quantal behavior of billiards is fascinating also; for example, the eigenfunctions of polygonal billiards were recently investigated by Amar, Pauri, and Scotti [17,18]: these authors discuss the cases in which such eigenfunctions can or cannot be represented as superpositions of plane waves. Their quantum mechanics has also been studied experimentally [19].

In this paper we show an entirely unexpected feature of the eigenstates of quantum billiards, namely, scarring by families of stable periodic orbits. Moreover, some of these eigenstates turn out to be scarred by narrow families of periodic orbits which *do not exist in the billiard itself but only in some neighboring pseudointegrable triangle.* We call such trajectories, when embedded in the ergodic billiard, *ghosts* of periodic orbits [20]; and we say that two triangles are "neighbors" if they are geometrically very similar. In this way an ergodic triangle can be the neighbor of an integrable one. Interestingly, our ghosts of periodic orbits are not dynamically acceptable trajectories of the ergodic billiard. We show that the ghosts scarring our states have a far more complicated phase space structure than the simple orbits supporting the bouncing ball states of the stadium. Some other states are shaped by larger families of "actual" orbits, i.e., not ghosts. We present eigenfunctions whose scarred nature is hidden in configuration space, but becomes evident in phase space. The phase space picture of the orbit supporting the state is very simple. However, we do not see any obvious link to states analogous to the bouncing ball states of the stadium billiard. In particular, we cannot identify the "slow" and "fast" variables needed for the Born-Oppenheimer approximation. In both cases, that is, ghosts and "actual" orbits, scars become evident when the state is viewed in the phase space (via the Husimi representation). We display our states in the phase space which is most proper to billiards, namely, the space of Birkhoff variables.

This paper is organized as follows: in the next section we review the method we used for producing the "quantum Poincaré section" of a given eigenfunction; in Sec. III we discuss our results for the scarring by ghosts; in Sec. IV we illustrate the scars produced by large families of periodic orbits; we present our general conclusions in Sec. V.

II. CLASSICAL AND QUANTUM POINCARÉ SECTIONS

In this section we briefly review the Birkhoff variables representation of the phase space of a billiard and explain how the quantum-mechanical description of the system can be embedded in it.

Leaving aside the unnecessarily complicated mathematical generalization involving concepts like "geodesic flows on a compact Riemannian manifold," a classical billiard is a very simple system. It consists of a free particle moving in a flat plane domain circumscribed by some (piecewise smooth) boundary. When the particle hits the boundary it is reflected according to the usual law: angle of reflection equals angle of incidence. It is apparent that the trajectory between two collision points is just a straight line: it does not carry any interesting information. One can effectively describe the dynamics of the billiard by keeping track only of the coordinate s of the collision points (i.e., the curvilinear distance from some reference point on the boundary) and the angle of reflection off the boundary θ (measured with respect to the inward normal to the boundary at the collision point). It is usual to normalize the length of the boundary to 1 and to use $\sin\theta$ as second coordinate. It is also customary to normalize the speed of the free particle to 1 so that $\sin\theta$ represents the local projection of the momentum of the particle along the boundary. The boundary is a closed curve, therefore $s=0$ and 1 are really the same point. In the Birkhoff variables picture the phase space of a billiard has the topology of a cylinder. It is easy to see that the missing parts of the trajectory (i.e., straight segments joining collision points) can be reconstructed exactly from the information encoded in the phase space just described. We see that in the case of classical billiards the phase space and the Poincaré surface of section coincide exactly if Birkhoff variables are employed.

One way to embed the quantum-mechanical solution of the problem in the Birkhoff variables phase space has recently appeared in the literature [21]. It consists of a method to produce a "quantum Poincaré section" (from now on to be denoted QPS) for each eigenstate of the system. The phase space of polygonal billiards has a very peculiar structure [22], and to the best of our knowledge ours is the first application of the method to the case of polygonal billiards. The Birkhoff variables describe just the collisions with the boundary, however, a quantum billiard is a Dirichlet problem and the wave function vanishes on the boundary of the billiard, i.e., on the classical collision points. Yet one can formally expand the wave function $\psi(x,y)$ near the boundary in a first-order Taylor series along the local inward normal to the boundary. In practice, for each point s of the boundary one evaluates the wave function on some interior point close to it. The interior point is to be determined by moving away from the boundary along the inward normal by a distance considerably shorter than the de Broglie wavelength of the eigenstate. In this way for each eigenstate $\psi_n(x,y)$ we obtain a corresponding reduced wave function $\tilde{\psi}_n(s)$ defined on the [0,1] interval. The Birkhoff variables, which provide the most natural phase space in the classical case, describe the motion just in the proximity of the boundary where the interesting dynamics take place; therefore the reduced eigenfunction yields the most proper quantum-mechanical phase space approach to the problem. A two-dimensional phase space portrait of $\tilde{\psi}_n(s)$ can be obtained by considering its Husimi transform,

$$\mathcal{H}_{\tilde{\psi}_n}(s,\kappa)=|\langle s,\kappa|\tilde{\psi}_n\rangle|^2 , \tag{1}$$

where $|s,\kappa\rangle$ is a coherent state centered on s in configuration space and on $\kappa=p/\hbar$ in wave number space. Its coordinate space representation is

$$\langle s'|s,\kappa\rangle=\left[\frac{1}{\pi\sigma^2}\right]^{1/4}\exp\left[-\frac{(s'-s)^2}{2\sigma^2}+i\kappa(s'-s)\right] . \tag{2}$$

The dispersions in position and wave number are $\sigma/\sqrt{2}$ and $1/\sqrt{2}\sigma$, respectively. We must also consider the fact that our reduced one-dimensional configuration space is topologically equivalent to a circle. While this can be formally done by introducing the notion of a "periodic coherent state" [21], the same aim can be achieved more simply by employing a "cut and paste" procedure. When a tail of the Gaussian envelope of the coherent state falls beyond one of the limits of the [0,1] interval we simply cut it and paste it back at the other end of the interval (see Fig. 1). It is apparent that one can use truncated Gaussian envelopes neglecting the infinite tails which would not contribute significantly to the overlap integral. Next we write $\kappa=\kappa_t\kappa_n$, where κ_n is the wave number of a given eigenstate and $-1<\kappa_t<1$. The new variable κ_t is the tangential component of the momentum along the boundary and it is to be associated to the second Birkhoff variable $\sin\theta$, which has the same physical meaning. We will plot the Husimi transform as a function of s and κ_t. We just need to evaluate the following integral:

$$\mathcal{H}_{\tilde{\psi}_n}(s,\kappa_t)=\left|\int_0^1 ds'\langle s,\kappa_t\kappa_n|s'\rangle\tilde{\psi}_n(s')\right|^2, \quad (3)$$

where the coherent state is understood in its truncated-tail version and with the "cut and paste" procedure described above. In Eq. (3) the dispersion coefficient σ should be chosen so as to have symmetric dispersion: $\Delta\kappa/2\kappa_n=\Delta s$. The integration can be reduced to a pair of one-dimensional real integrals. As can be seen from Fig. 1, even at high energies the reduced wave function $\tilde{\psi}_n(s)$ does not oscillate rapidly and is fairly amenable to numerical treatment. We also normalized the result so as to make the maximal height of the QPS equal to 1. Finally, it is worth noting that since we rescaled the length of the perimeter to unity, every wave number calculated from a billiard of different size must be properly rescaled according to Weyl's law [1]. This procedure was recently applied to the case of a smooth boundary billiard with mixed phase space, in which islands of stability coexist with chaotic regions [21]. The results showed good agreement with the classical Poincaré surfaces of section.

III. SCARRING BY GHOSTS

In the present section we introduce the triangular billiards we investigated and display the results of our calculations.

We originally observed the existence of scarred states in a triangle with angles which are irrational (within the precision of the computer) multiples of π. This "irrational" triangle, henceforth denoted triangle C, was obtained by distorting one of the angles of an equilateral triangle from $\pi/3$ to $\pi/\sqrt{10}$, leaving the ratio of the height to the basis untouched. This classical billiard is generally conjectured to be ergodic. We calculated the eigenstates of triangle C using a quantization scheme which is described in an earlier paper [5].

The 875th eigenstate was the first strongly scarred eigenstate we observed. A contour plot of the probability density is displayed in Fig. 2 along with one of the ghosts of periodic orbits which belongs to the family scarring the eigenstate. The eigenstate of Fig. 2 was expanded on a truncated Hilbert space basis of 3000 orthonormal vectors ordered in energy [5]. The components of the state are plotted in Fig. 3. The reliability of the numerical result is remarkable: it is evident from this figure that very high energy basis vectors do not contribute significantly to the eigenstate, therefore a truncated basis does not induce a large error in the calculation. In the small triangle in the upper right corner of Fig. 2 we display the region of configuration space spanned by the family of ghosts. Remarkably, the periodic orbit of Fig. 2 does not reside in triangle C. After the discovery of the scar we investigated many other energy ranges of the spectrum and found that states showing essentially the same con-

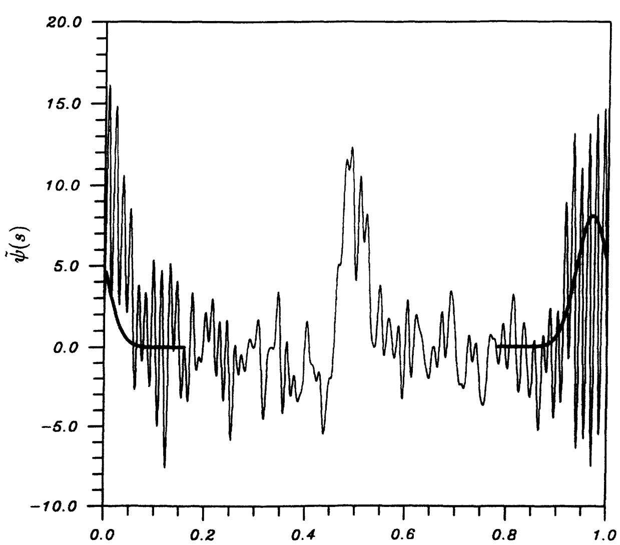

FIG. 1. The reduced function for the 875th state of triangle C along with the Gaussian envelope of a coherent state illustrating the "cut and paste" procedure (see text).

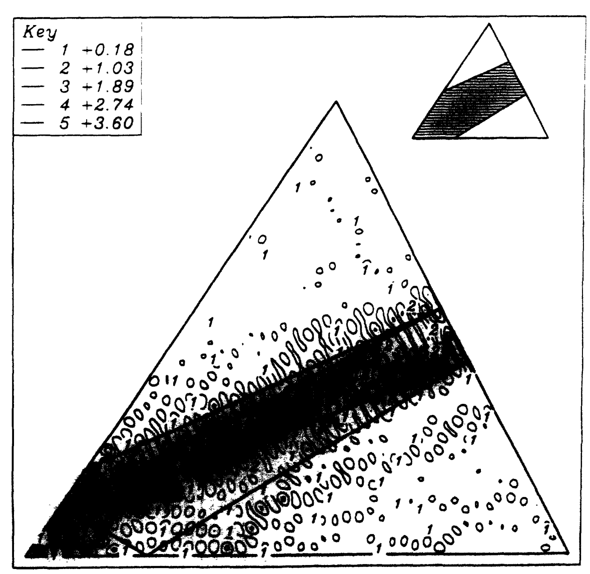

FIG. 2. Contour plot of the modulus squared of the 875th eigenfunction of triangle C along with a *ghost* of a periodic orbit. The scarred nature of the wave function is apparent. The small triangle shows the region spanned by the family of periodic orbits.

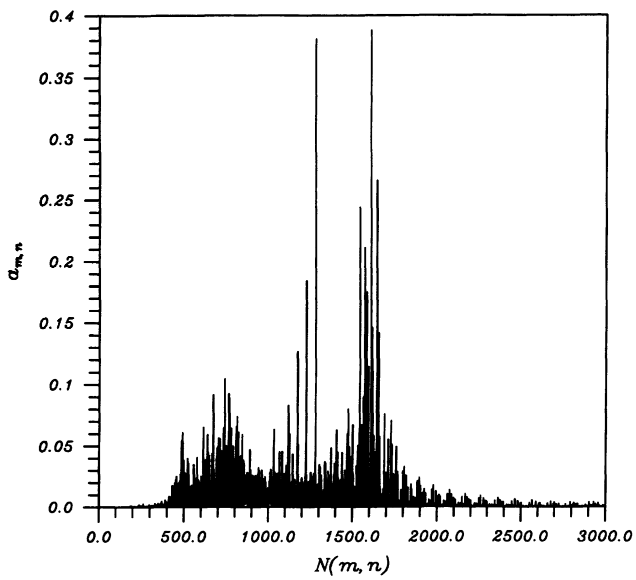

FIG. 3. Components $a_{m,n}$ of the scarred eigenstate along the basis vectors considered. The contribution from high energy basis vectors is negligible. This makes the numerical result reliable.

tour pattern as Fig. 2 are ubiquitous. We detected them from energies corresponding approximately to the 400th eigenstate up to energies close to the 1200th level; in that interval they account for roughly 3% of the spectrum (in a less systematic search, however, we also detected scarred states at very low energy, namely, around the 40th eigenstate). The high energy scarred states seem to show a slightly sharper localization both in configuration space and in phase space (their phase space picture will be illustrated below) than their low energy cousins. It is worth remembering that in the case of billiards the correspondence between scarred eigenstates and periodic orbits is purely geometrical. In a classical billiard the infinitely high walls corresponding to the reflecting boundary imply that the trajectory of the particle is independent of its energy. This is why the speed of the particle can be freely normalized to one. As a consequence, a single periodic orbit can scar many eigenstates of widely different energies.

The very existence of periodic orbits with a nontrivial structure is doubtful in the case of irrational billiards. This can be seen by considering the map under reflection for the angle that a trajectory makes with a fixed reference direction [23].

$$\phi \rightarrow \phi' = 2\alpha_j - \phi \ , \tag{4}$$

where α_j is the angle between the side upon which the collision takes place and the reference direction. It is easy to see that a sequence of n reflections will yield the following map:

$$\phi \rightarrow \phi' = 2\sum_j \sum_l (-1)^{l+1}\alpha_j + (-1)^n \phi \ , \tag{5}$$

where l is the order of the collision and the second summation includes a term only if the lth collision is on the jth side. The periodicity in the angle of the trajectory (which, by the way, would not guarantee the existence of a periodic orbit) is connected to the number-theoretical properties of the angles of the polygon: the numbers α_j/π must be linearly dependent on the rationals. Indeed, some alternative and improbable cases can also happen. For some simple orbits (as we shall see later) it can be that all the l summations in Eq. (5) vanish independently. Alternatively it can happen that the result of the double summation in the same equation equals $2\phi + 2m\pi$: this case would yield periodicity in the case n is odd. For an even number of reflections (which is the case for the periodic orbit of Fig. 2) the periodicity of the angle does not depend on the initial angle but solely on the pattern of collisions. In conclusion, it is evident that the existence (or otherwise) of periodic orbits in irrational billiards is not a trivial question.

After some discouraging numerical searches for at least some closely recursive orbit localized in the region where the wave function of Fig. 2 is peaked, we started looking for periodic orbits in some triangle with angles in rational ratio with π which approximates triangle C. The periodic orbit of Fig. 2 was immediately discovered in a "rational" triangle, henceforth denoted "triangle D," which was obtained by a two-decimal-digit approximation of the angles of triangle C. These two triangles are displayed in Fig. 4. The angles at the basis of triangle D are

$$\alpha_D = \tfrac{8}{25}\pi \ , \qquad \beta_D = \tfrac{7}{20}\pi \ . \tag{6}$$

The phase space of a rational polygonal billiard is structured by invariant surfaces of high *genus*. One can easily calculate [24] the *genus* of the invariant surface of trian-

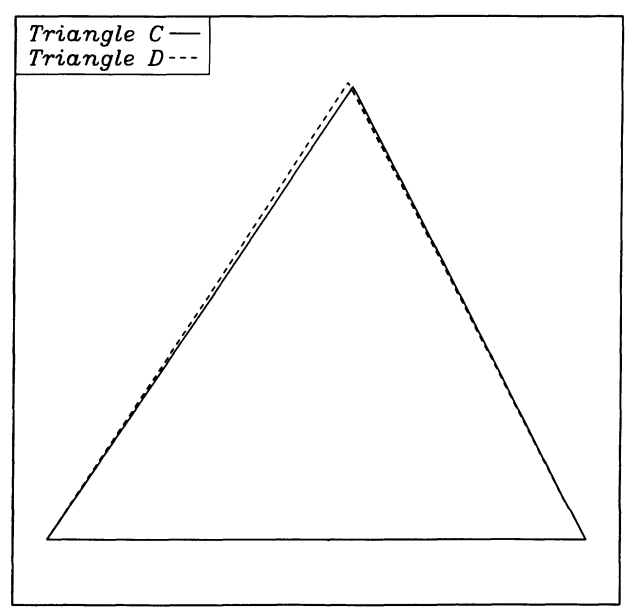

FIG. 4. The two triangles considered: triangle C (solid line) vs triangle D (dashed line). Notice that triangle D is slightly larger.

gle D to be $g_D = 96$. A system characterized by such a phase space structure is called "pseudointegrable" [22]. A simple analysis shows that the ghost orbit is exactly periodic in triangle D. Obviously, in correspondence of this orbit there exist a whole family of periodic trajectories sharing the same pattern of reflections. Their branches are parallel to the ones of the orbit displayed in Fig. 2. The initial conditions that lead to the orbits which generate ghosts, when used in triangle C, yield trajectories which stray rapidly from the region in which the eigenfunction is scarred. These trajectories do so in a time shorter than the period of the ghosts. The same holds for other initial conditions in the phase space neighborhood of the ones which generate the exact periodic orbits of triangle D.

However, one does not have to rely solely on visual evidence to implicate this family in the scarring of this eigenfunction. The best evidence for the connection between this family and the scar in triangle C emerges from the Birkhoff variable representation of the eigenstate. It is easy to see that since all the trajectories of the family share the same sequence of angles (but different collision points), the Birkhoff variables picture of such a family will consist of straight line segments parallel to the basis of the cylinder. The result of the comparison between the QPS of the eigenstate (thin contour lines) and the classical Poincaré section for the family of orbits (thick straight segments) is displayed in Fig. 5. The match is striking. It is noteworthy that more than 70% of the

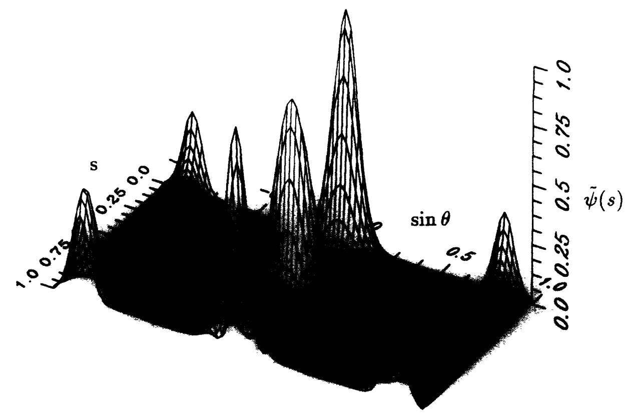

FIG. 6. Three-dimensional version of the quantum Poincaré section of Fig. 5. The dominant nature of the peaks is clearly revealed by the three-dimensional display.

volume defined by the surface determining the QPS is actually enclosed in those regions which are bounded by the contours plotted in Fig. 5. The dominance of such regions can be well appreciated by viewing the three-dimensional picture of the same QPS (Fig. 6). For comparison we show, in Fig. 7, the QPS of a typical wave function uniformly spread in configuration space: the stark difference from Fig. 5 is apparent. We observed essentially the same features as in Fig. 5 in all the QPS's for the other scarred eigenstates. We also studied the

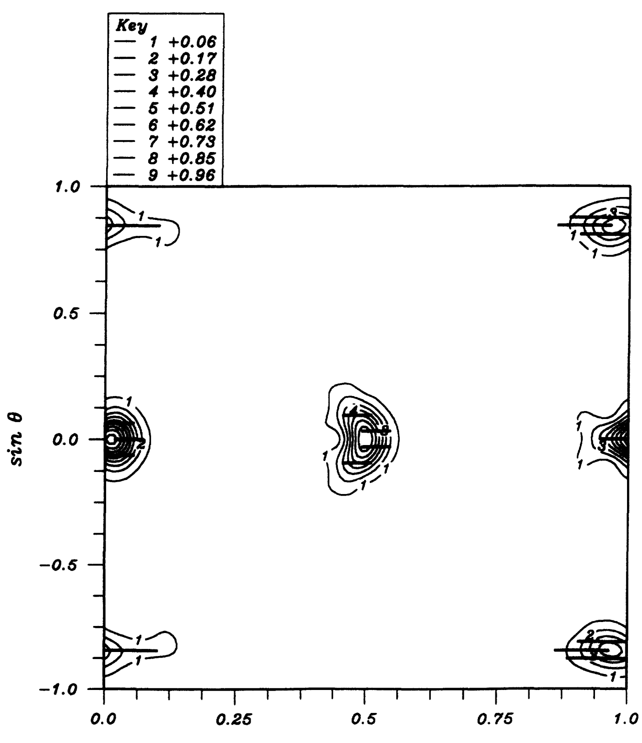

FIG. 5. Phase space representation of the scarred eigenstate of Fig. 2. The straight segments are the phase space picture of the family of ghosts scarring the state. The match between the two patterns is perfect.

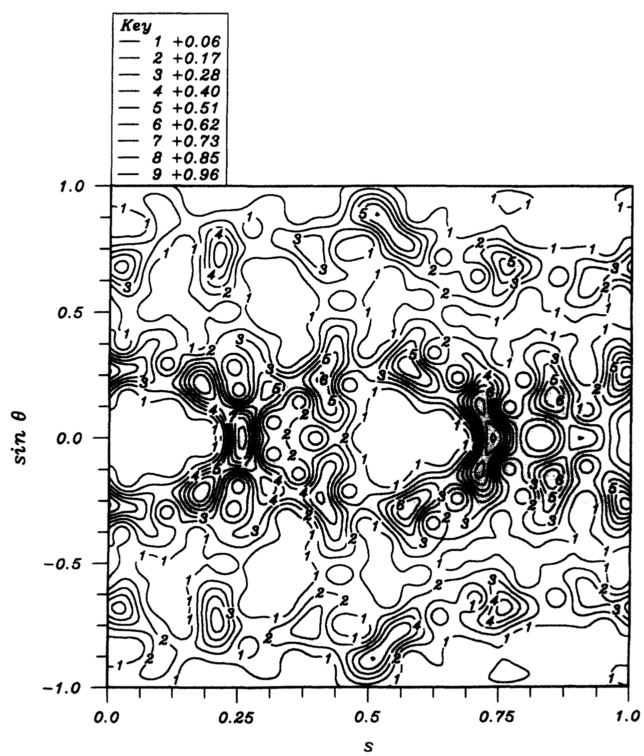

FIG. 7. Quantum Poincaré section of a generic eigenfunction of triangle C. The difference from the contour pattern of Fig. 5 is evident.

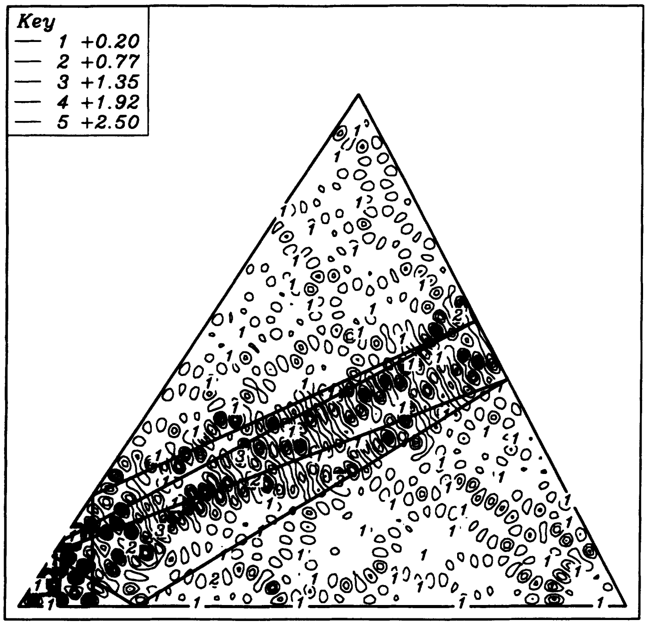

FIG. 8. Contour plot of the 884th eigenfunction of triangle D. A periodic orbit is also plotted. The scar is easily detectable. Some minor structures are also present. However, they do not influence significantly the quantum Poincaré section.

eigenstates of triangle D and, of course, we found some states which are scarred along the same family of periodic orbits (Fig. 8). One can notice immediately that the scar in triangle D is not as marked: other, minor, structures are present. However, its QPS shows the same features as in Fig. 5, leading one to argue that the scar is a "robust" feature of the eigenfunction and therefore can resist some small perturbation of the boundary.

Another argument supporting the scarring by ghosts of periodic orbits comes from the comparison of the energies of scarred eigenstates in triangles C and D. As it can be appreciated from Fig. 4, triangle D is slightly larger than triangle C. According to Weyl's law this implies that the nth energy level of triangle D must be approximately 1% smaller than the corresponding eigenvalue in triangle C. Our results satisfy this requirement. Yet the two eigenstates of Figs. 5 and 8 have approximately the same energy: They differ by less than 0.2%. The same holds for the other scarred eigenstates we observed. The similarity both in the contour pattern and in the phase space representation leaves little doubt that as triangle D is slightly distorted into triangle C the eigenfunction of Fig. 8 (i.e., the 884th eigenstate of triangle D) evolves into the wave function of Fig. 5 (i.e., the 875th eigenstate of triangle C). This evolution must be accompanied by multiple level crossings [25].

IV. SCARRING BY FAMILIES OF PERIODIC ORBITS

Eigenstate scarring in triangular billiards is not only due to ghosts: We also observed some states scarred along large families of regular periodic orbits. In contrast to the case of ghosts, such exceptional families of orbits reside in the same billiard as the scarred eigenfunctions themselves. Figures 9 and 10 display two sample states for triangles C and D, respectively, along with a representative of the family of periodic orbits that we believe is scarring the state. Many other states with a similar diffuse contour pattern and a similar phase space QPS were also observed in many energy ranges. This diffuse nodal structure in configuration space is ordinarily a sign of the absence of scarring, yet scarring becomes obvious once a phase space picture is produced. Both eigenstates of Figs. 9 and 10 are scarred by "actual" periodic orbits which exist in the domain proper to each eigenstate. In Fig. 11 we show the QPS for the 862nd eigenstate of triangle C. The result for triangle D is essentially the same, since the structure of the periodic orbits is very similar. The fraction of volume contained in the regions enclosed by the contour lines displayed in the figure is approximately 90%.

Note that, in contrast to the ghosts from neighboring triangles, we are now considering two different exact periodic orbits, each of them existing in one of the triangles. The periodicity in the angle of the trajectory under the pattern of reflections can be easily checked analytically. It does not depend on the linear dependence on the rationals of the angles of the billiard since now all the l summations of Eq. (5) vanish independently. In order to prove the periodicity in configuration space, we plotted the distance between an initial point and a final point on some side of the triangle as a function of the angle of the orbit. The existence of a zero (which could be evaluated numerically with great accuracy) was apparent. This implies the existence of an exact periodic orbit.

The two states of Figs. 9 and 10 are obviously in

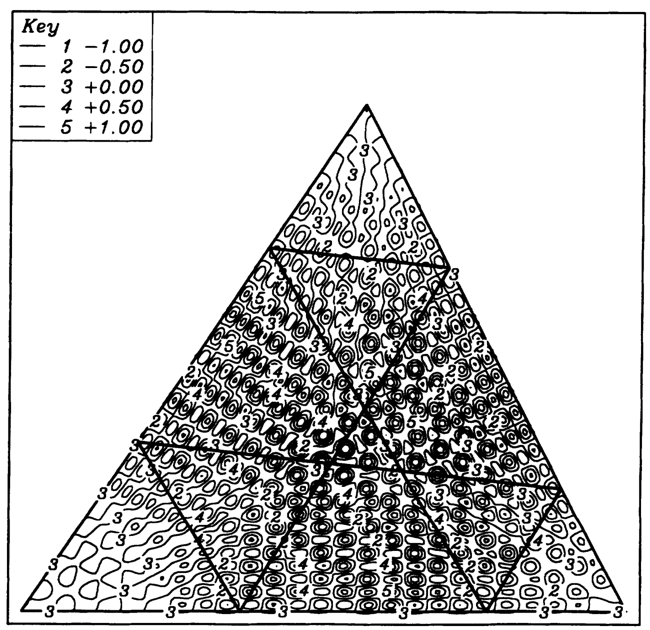

FIG. 9. Contour plot of the 862nd eigenstate of triangle C. A periodic orbit belonging to the family which scars the state is also plotted.

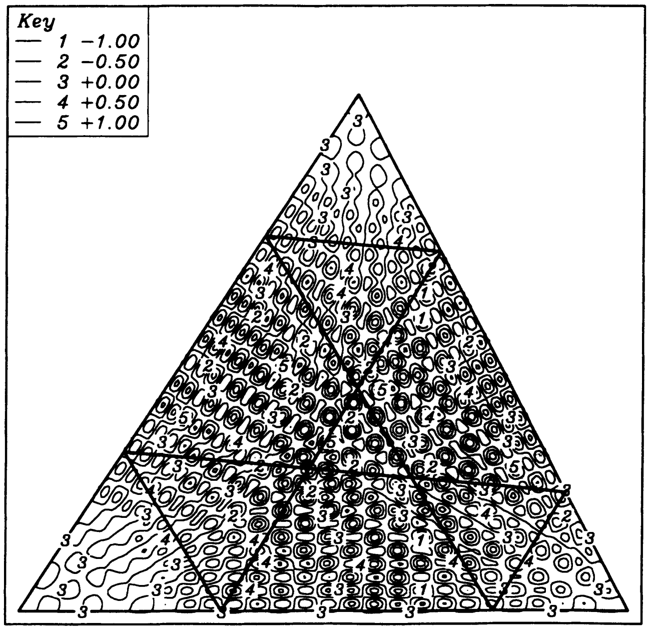

FIG. 10. Contour plot of the 862nd eigenstate of triangle D. A periodic orbit belonging to the family which scars the state is also plotted.

correspondence with each other, in the sense that it is reasonable to assume that under some continuous deformation of, say, triangle D into triangle C, the state of Fig. 10 (an eigenstate of triangle D) must continuously transform into the state of Fig. 9, which is an eigenstate of triangle C. The same correspondence was considered for the states scarred by ghosts and their corresponding states in triangle D. In the present case, the periodic orbits also continuously transform into new periodic orbits of the new domain, without a significant change in their phase space structure. As a consequence both eigenstates are the 862nd state for their triangle; in contrast with the case of scarring by ghosts their energies scale according to Weyl's law.

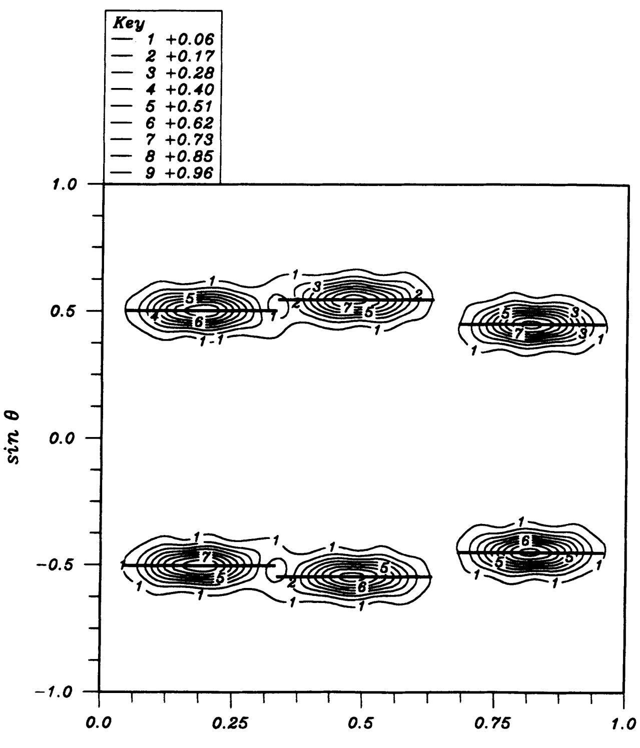

FIG. 11. Quantum Poincaré section for the same state as in Fig. 9. The scarring by the family of periodic orbits is evident (straight segments).

V. CONCLUSIONS

In the present paper we have shown strong evidence for the existence of scarred eigenstates in nonchaotic billiards.

All present theories of scars assume that the orbit scarring a given eigenstate is isolated. Such is not the case for the scars discussed in this paper. All the trajectories scarring our eigenfunctions are stable and belong to families of periodic orbits. Even though the influence of families of parallel orbits on the shape of the wave functions had already been detected in the stadium billiard, our orbits show a more complex phase space picture than the bouncing ball orbits.

We have also shown that orbits which belong to some neighboring domain can scar the eigenstates of a billiard. We call such trajectories ghosts of periodic orbits. This feature is peculiar to polygonal billiards and is due to sudden changes in the nature of the classical motion versus *adiabatic* changes in the quantum regime. This *quantum adiabaticity* poses serious problems for the definition of a precise quantum-classical correspondence principle. An infinitesimal change in the angles of a polygonal billiard is sufficient to destroy any invariant phase space surface. It can be argued that, under some infinitesimal perturbation of the boundary, the eigenfunction and eigenvalues of a billiard cannot change abruptly. We have shown that this is indeed correct: under a change of the order of 2% in the domain, the quantum mechanics of the system is still strongly influenced by the neighboring pseudointegrable billiard and by the complicated structure of its phase space. However, it is clear that the somewhat artificial nature of polygonal billiards must be responsible for this discrepancy. It is certainly true that in nature one never has completely impenetrable walls and infinitely sharp corners. The classical motion is extremely sensitive to the question of the irrationality of the angle of the polygon. However, such questions are irrelevant in the quantum regime and, in a way, also in a *physical* outlook on classical billiards [26].

The eigenstates studied in this paper are not the only ones which contain some underlying classical structures. We observed many other states, in both triangles C and D, that are influenced by classical trajectories. Unfortunately in triangular billiards it is not easy to find families of periodic orbits which are both simple and confined in a small region of configuration space. In the case at hand the obvious resonances in the angles of triangle D favor a proliferation of families of periodic orbits. This makes the task of identifying the family responsible for scarring extremely difficult for cases in which the geometry of the scar is not as favorable as in the ones displayed in this paper.

The present paper raises many interesting questions. Are the scars along ghosts of periodic orbits going to be healed by the semiclassical limit? Are scars a robust feature of the eigenfunction? What happens to a scar along some unstable periodic orbit when the domain is slightly perturbed? In our view the stadium billiard, as the very birthplace of scars, would be the most proper arena for such a study, and these studies are in progress in our group.

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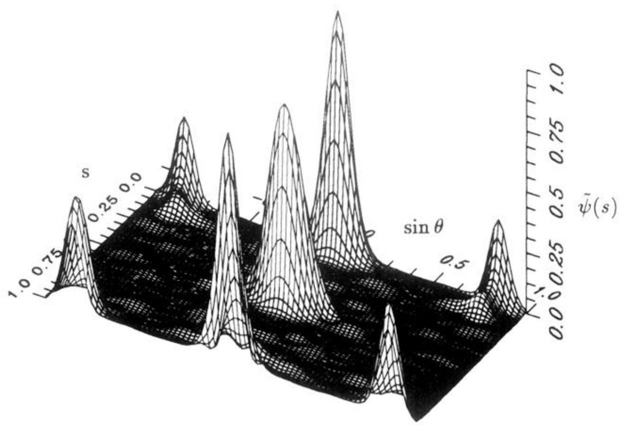

FIG. 6. Three-dimensional version of the quantum Poincaré section of Fig. 5. The dominant nature of the peaks is clearly revealed by the three-dimensional display.